An introduction to Construction Schemes

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CENTRO DE CIENCIAS MATEMÁTICAS

$\mathcal{F} \subseteq [\omega_1]^{<\omega}$ is a construction scheme of type $\{(m_{k+1}, n_{k+1}, r_{k+1})\}_{k\in\omega}$ if there is a decomposition $\mathcal{F} = \bigcup_{k\in\omega} \mathcal{F}_k$ such that for all $k\in\omega$:

1) $\mathcal{F}_0 = [\omega_1]^1$
2) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$
3) If $F \in \mathcal{F}_k$, then $|F| = m_k$
4) If $F, E \in \mathcal{F}_k$, then $F \cap E \sqsubseteq F, E$

5) For all $F \in \mathcal{F}_{k+1}$, there are $\{F_i \mid i < n_{k+1}\} \subseteq \mathcal{F}_k$ and $R(F)$ such that:

a) $F = \bigcup_{i < n_{k+1}} F_i$

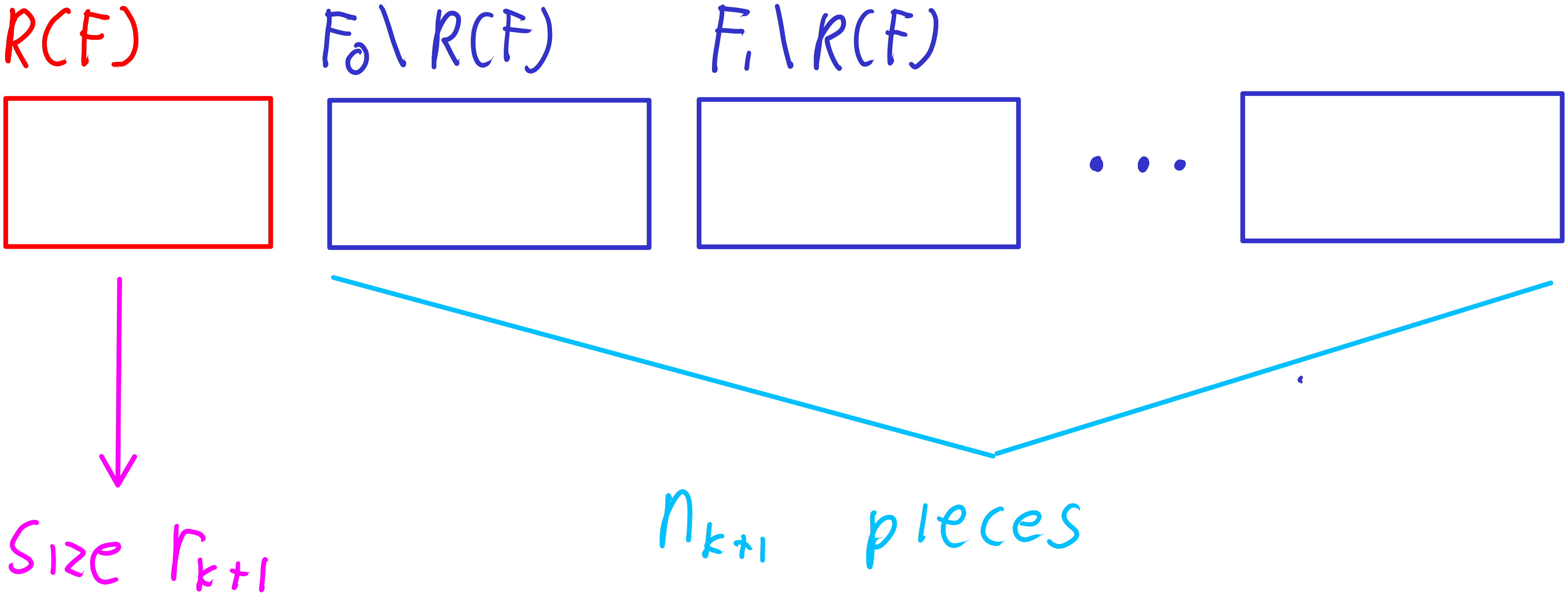
b) $\{F_i \mid i < n_{k+1}\}$ is a Δ-system with root $R(F)$

c) $|R(F)| = r_{k+1}$

d)

$$R(F) < F_0 \setminus R(F) < F_1 \setminus R(F) < \ldots < F_{n_{k+1}-1} \setminus R(F)$$

Each $F \in \mathcal{F}_{k+1}$ looks like this:



Theorem (Todorcevic)

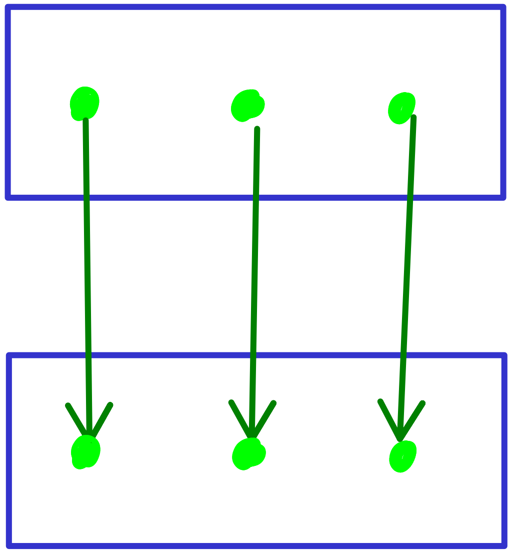
Construction Schemes exist (for any type)

Def

Let $E, F \in [\omega_1]^{<\omega}$ of the same size. Denote by φ_{EF} the only increasing bijection from E to F.

E

F



Applications of construction schemes are often obtained as follows:

Assume we want to build certain kind of structure of size ω_1

Fix $\mathcal{F}$ a construction scheme

1) Find $\mathbb{P}$ a forcing consisting of finite approximations of the desired structure

2) For every $F \in \mathcal{F}$, find some $p_F \in \mathbb{P}$ such that:

a) If $E \in \mathcal{F}$ and $E \subseteq F \Rightarrow p_F \leq p_E$

(p_F is a better approximation than p_E)

b) If $E, F \in \mathcal{F}_k$, then p_F and p_E are compatible. Moreover, they must be "coherent".

3) Look at the structure generated by $\{\rho_F \mid F \in \mathcal{F}\}$

4) ... hope for the best!

We will construct a gap and an Aronszajn tree with construction schemes. I learn this constructions in the thesis of Fulgencio Lopez.

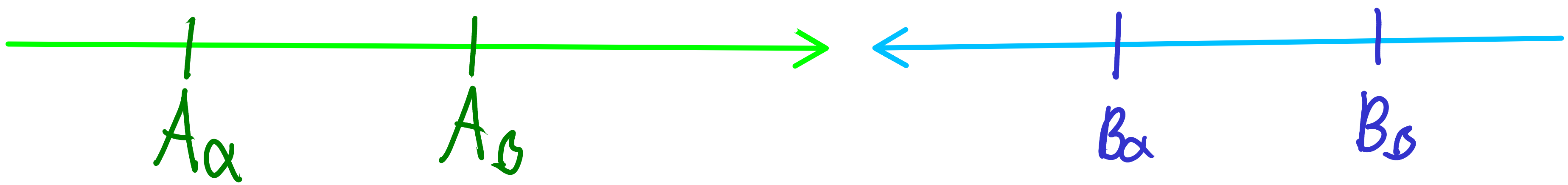
Constructing a gap

Let $A, B \subseteq \omega$

We say that $A \subseteq^* B$ (A is almost contained in B) if $A \setminus B$ is finite

1) We say that $\mathcal{G} = (\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle B_\alpha \rangle_{\alpha<\omega_1})$ is a pregap if for every $\alpha < \beta$:

$$A_\alpha \subseteq^* A_\beta \subseteq^* B_\beta \subseteq^* B_\alpha$$



2) A pregap $\mathcal{C} = (\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle B_\alpha \rangle_{\alpha<\omega_1})$ is a gap if there is no C such that:

$$A_\alpha \subseteq^* C \subseteq^* B_\alpha \text{ for all } \alpha < \omega_1$$

2) A pregap $\mathcal{G} = (\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle B_\alpha \rangle_{\alpha<\omega_1})$ is a **gap** if there is no C such that:

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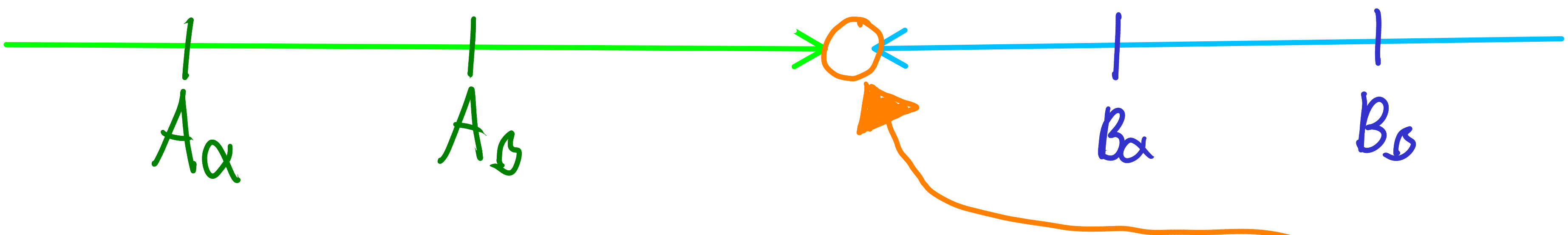


Nothing can be placed here

Gaps are very concrete example that $P(\omega)/\mathrm{fin}$ is not complete

Theorem (Hausdorff)

There is a gap

Theorem (Hausdorff)

There is a gap

The usual proof is by a recursion of size ω_1. We will prove the Theorem using construction schemes

Knowledge of gaps is fundamental when we are trying to embedd structures into $P(\omega)/\mathrm{fin}$

Gaps represent possible obstructions we may encounter.

For example, assume we have a linear order $L = \{p_\alpha \mid \alpha \in \omega_2\}$ and we want to construct an embedding $F : L \to P(\omega)/fin$ (of course, we are in a world where CH fails).

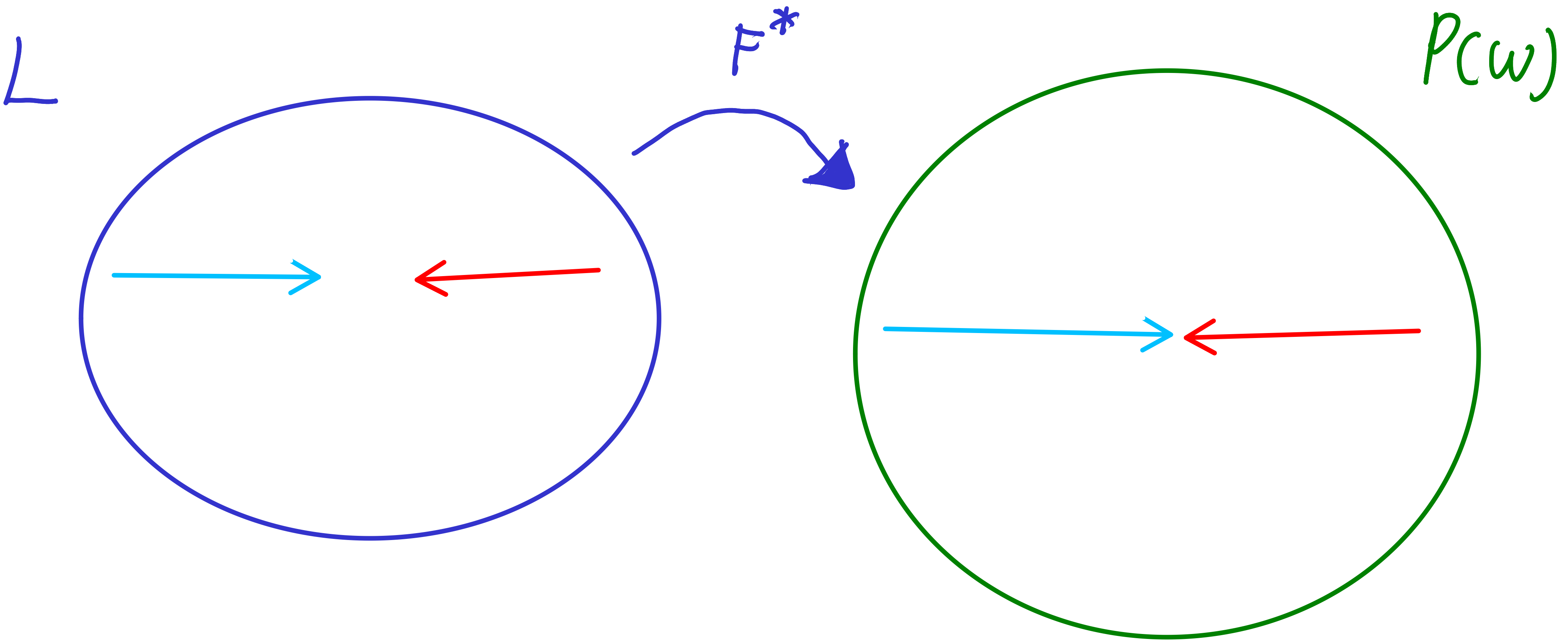
We try to define F recursively.

It could happen that in the ω_1 step, we have a partial embedding

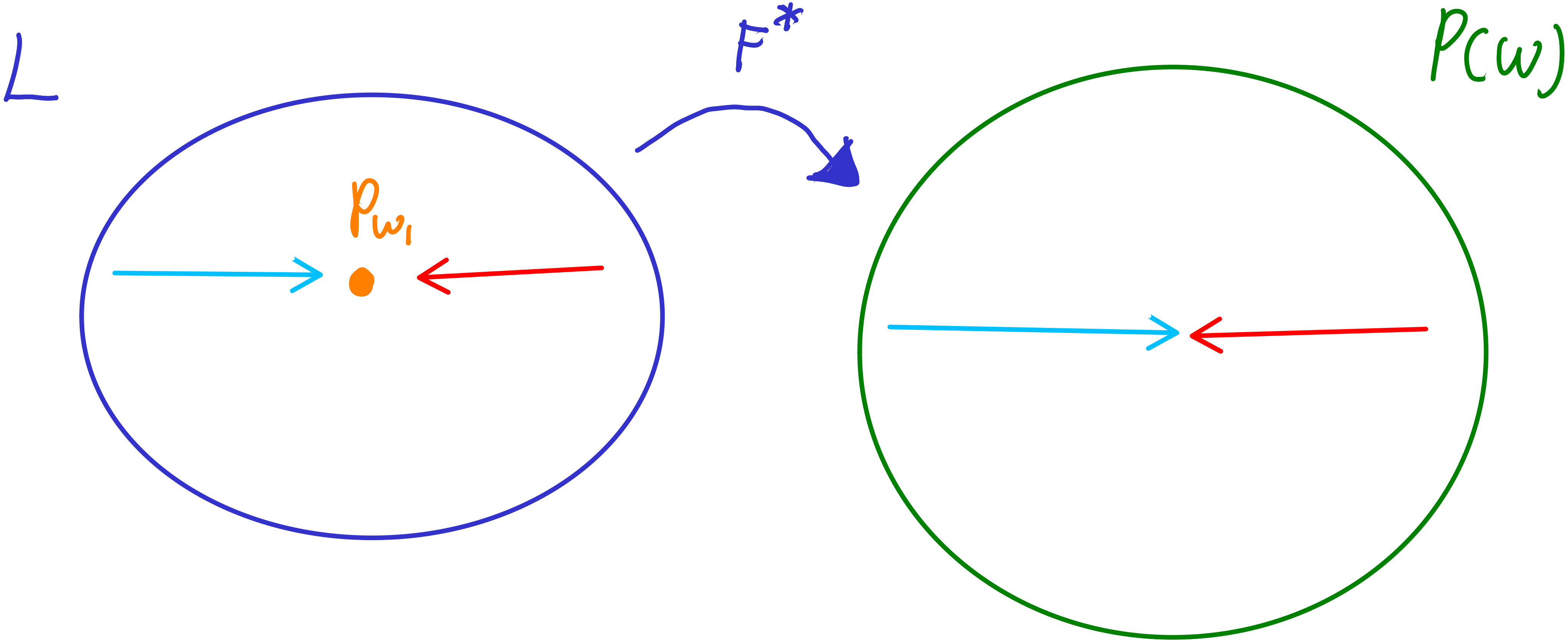
$$F^*: \{p_\alpha \mid \alpha < \omega_1\} \longrightarrow P(\omega)$$

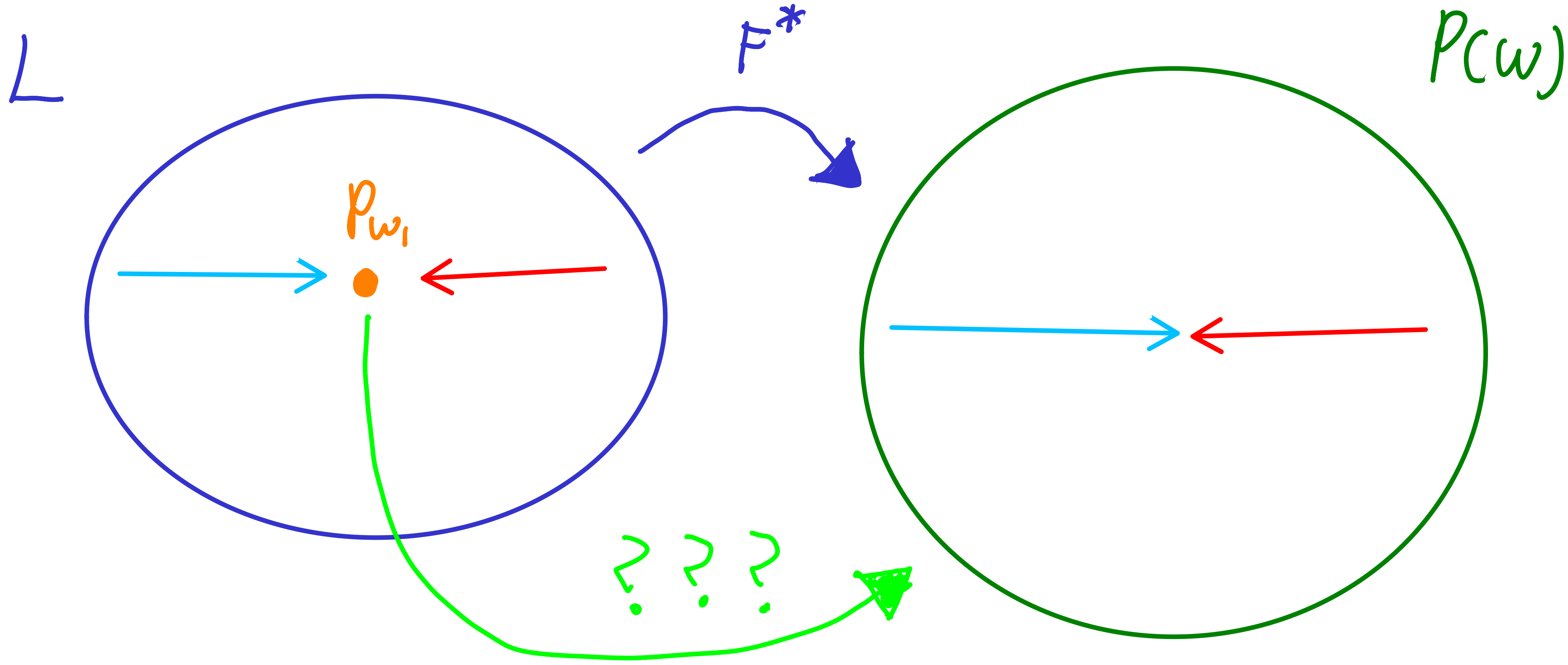
such that:

1) There are $X, Y \in [\omega_1]^{\omega_1}$ such that $(F^*[\{p_\alpha | \alpha \in X\}], F^*[\{p_\alpha | \alpha \in Y\}])$ is a gap



2) $p_\alpha < p_{w_1} < p_\beta$ for all $\alpha \in X$ and $\beta \in Y$





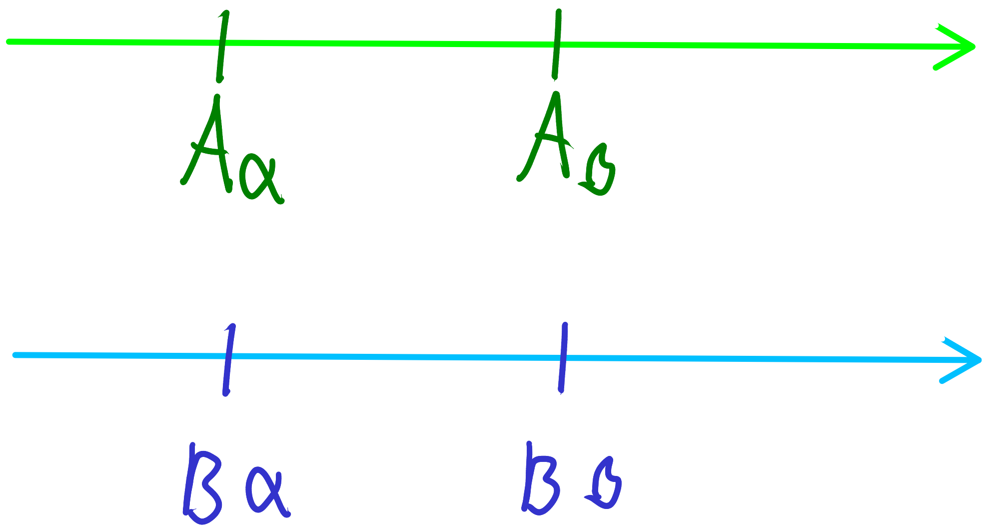
L
F^*
$P(w)$
p_{w_1}
???

In here, F^* can not be extended to an embedding, so our attempt to embedd L into $P(\omega)/\mathrm{fin}$ failed.

It will be convenient to reformulate the notion of gap

1) We say that $\mathcal{C} = (\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle B_\alpha \rangle_{\alpha<\omega_1})$ is a pregap if for every $\alpha < \beta$:

1) $A_\alpha \subseteq^* A_\beta$
2) $B_\alpha \subseteq^* B_\beta$
3) $A_\alpha \cap B_\alpha = \emptyset$



2) A pregap $\mathcal{C} = (\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle B_\alpha \rangle_{\alpha<\omega_1})$ is a gap if there is no C such that:

1) $A_\alpha \subseteq^* C$ for all $\alpha < \omega_1$

2) $B_\alpha \cap C$ is finite for all $\alpha < \omega_1$

2) A pregap $\mathcal{G} = (\langle A_\alpha \rangle_{\alpha<\omega_1}, \langle B_\alpha \rangle_{\alpha<\omega_1})$ is a gap if there is no C such that:

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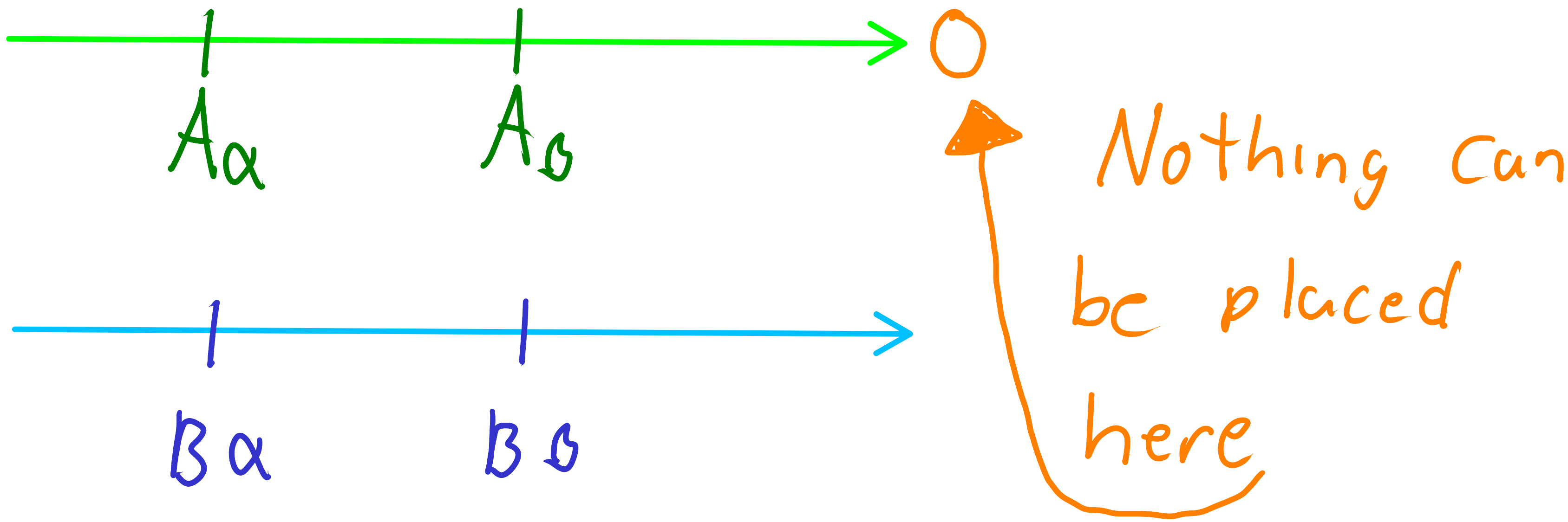


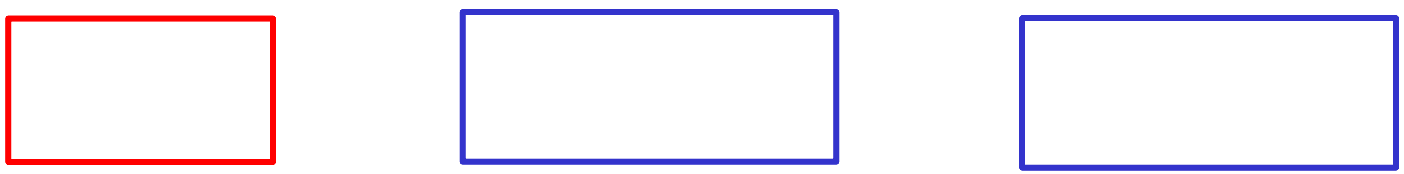
Both definitions of gaps are essentially the same.

From now on, every mention of "gap" refers to the second meaning

Fix $\mathcal{F}$ a $\langle (m_{k+1}, 2, r_{k+1}) \rangle_{k \in \omega}$ construction scheme

Fix $\mathcal{F}$ a $\langle (m_{k+1}, 2, r_{k+1}) \rangle_{k\in\omega}$ construction scheme

This means we are always gluing two thing of $\mathcal{F}_k$ to produce one in $\mathcal{F}_{k+1}$



Let $\mathbb{P}$ be the set of all $p=(X, n, \mathcal{A}_p, \mathcal{B}_p)$:

1) $X \in [\omega_1]^{<\omega}$ and $n \in \omega$

X will be referred as the domain of p ($X = \mathrm{dom}(p)$) and n as the height of p ($n = \mathrm{ht}(p)$)

Let $\mathbb{P}$ be the set of all $p = (X, n, \mathcal{A}_p, \mathcal{B}_p)$:

1) $X \in [\omega_1]^{<\omega}$ and $n \in \omega$

2) $\mathcal{A}_p = \langle A^p_\alpha \rangle_{\alpha \in X}$, $\mathcal{B}_p = \langle B^p_\alpha \rangle_{\alpha \in X}$

Let $\mathbb{P}$ be the set of all $p = (X, n, \mathcal{A}_p, \mathcal{B}_p)$:

1) $X \in [\omega_1]^{<\omega}$ and $n \in \omega$

2) $\mathcal{A}_p = \langle A_\alpha^p \rangle_{\alpha \in X}$, $\mathcal{B}_p = \langle B_\alpha^p \rangle_{\alpha \in X}$

3) $A_\alpha^p, B_\alpha^p \subseteq n$ for $\alpha \in X$

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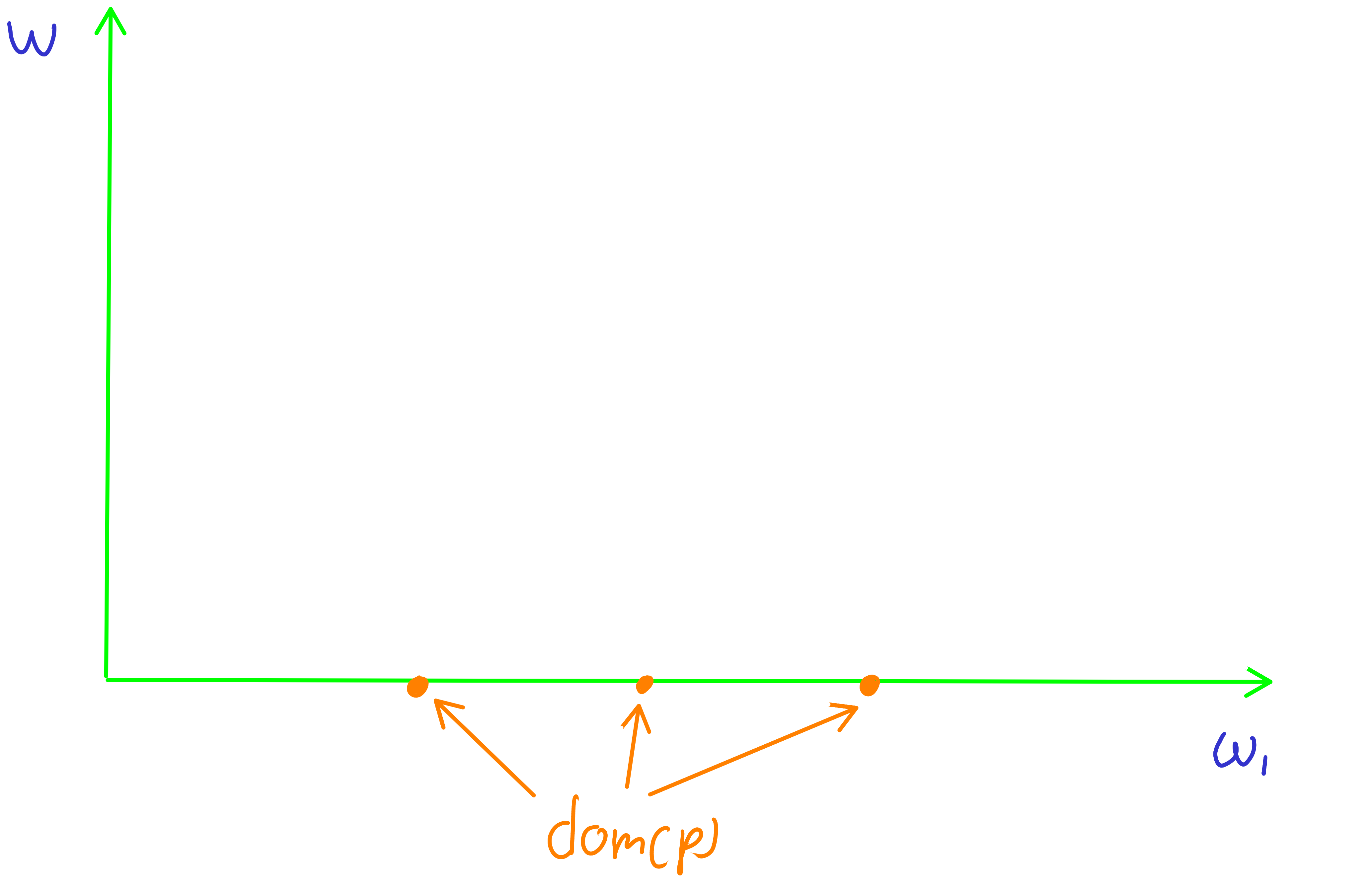
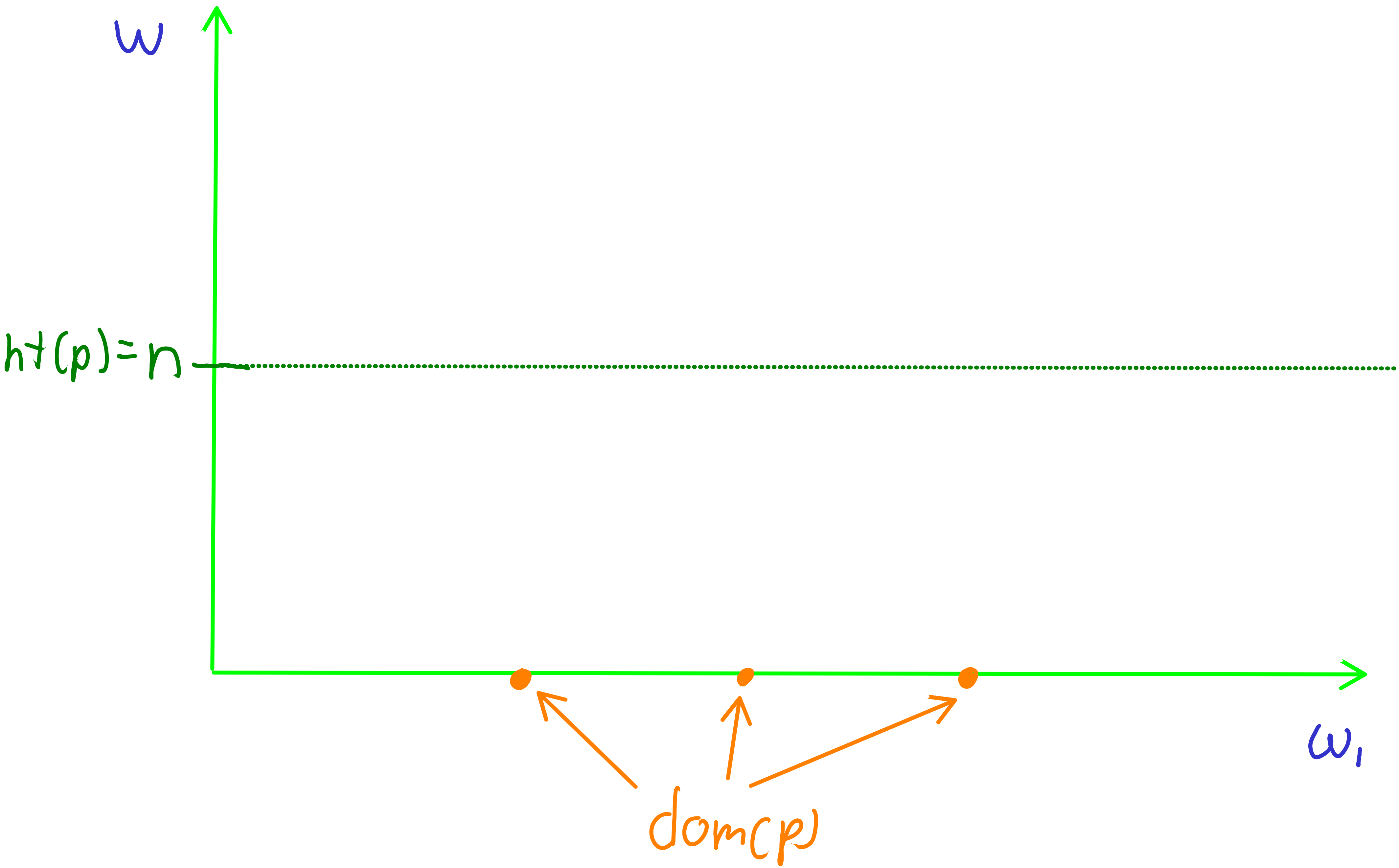
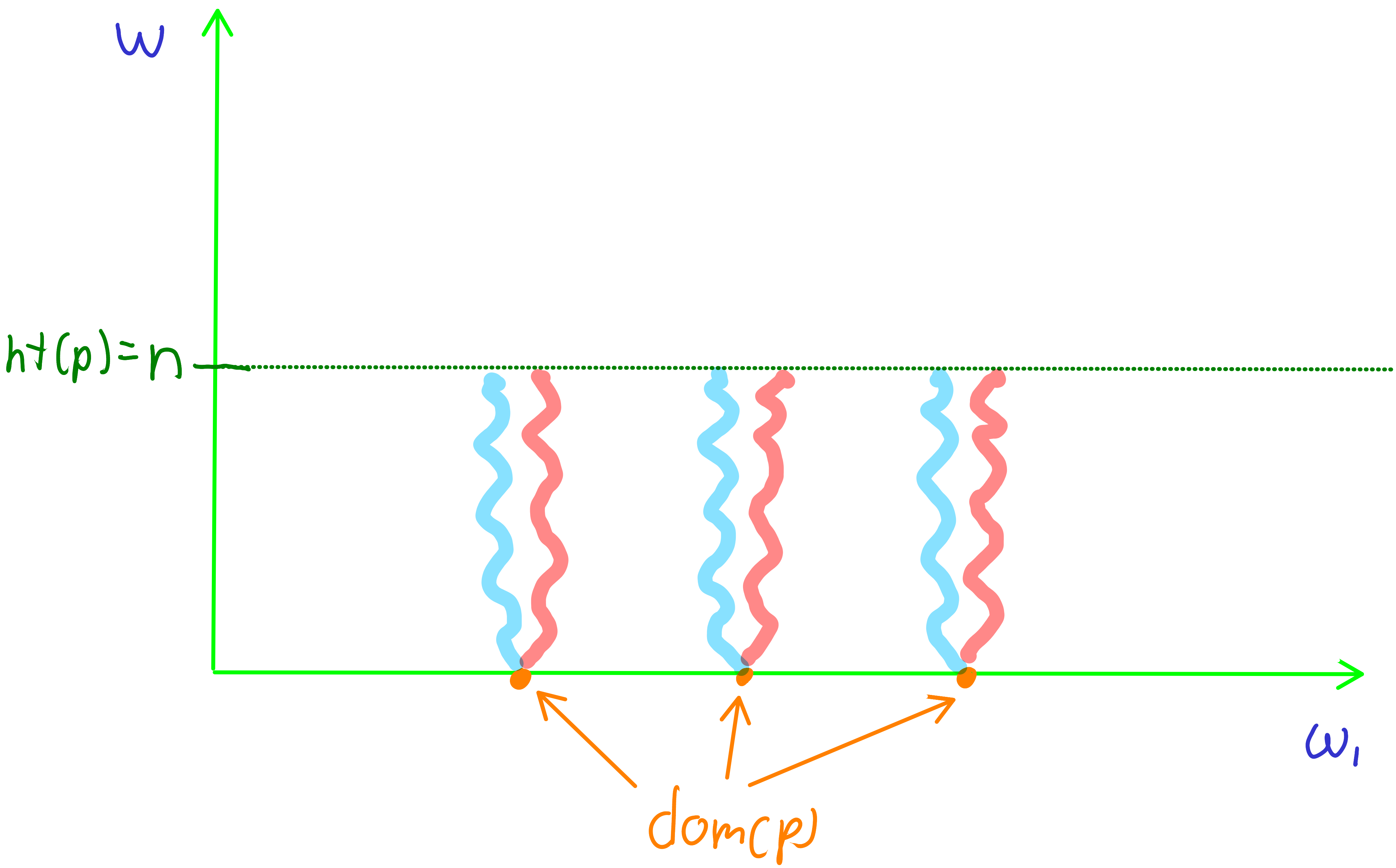
w
ω_1



w
ω_1
dom(p)



ω
ht(p) = n
ω₁
dom(p)



ω
ht(p)=n
ω₁
dom(p)

Let $p = (X_p, n_p, \mathcal{A}_p, \mathcal{B}_p)$ and $q = (X_q, n_q, \mathcal{A}_q, \mathcal{B}_q)$

Define $p \leq q$ (p is a better aproximation than q) if:

Let $p = (X_p, n_p, \mathcal{A}_p, \mathcal{B}_p)$ and $q = (X_q, n_q, \mathcal{A}_q, \mathcal{B}_q)$

Define $p \leq q$:

1) $X_q \subseteq X_p$ and $n_q \leq n_p$

2) If $\alpha \in X_q$, then:

a) $A_\alpha^q = A_\alpha^p \cap n_q$

b) $B_\alpha^q = B_\alpha^q \cap n_q$

3) If $\alpha, \beta \in X_q$ with $\alpha < \beta$, then:

a) $A_\alpha^p \setminus n_q \subseteq A_\beta^p$

b) $B_\alpha^p \setminus n_q \subseteq B_\beta^p$

Define $p \leq q$:

1) $X_q \subseteq X_p$ and $n_q \leq n_p$

2) If $\alpha \in X_q$, then:

$$A^q_\alpha = A^p_\alpha \cap n_q \quad \text{and} \quad B^q_\alpha = B^q_\alpha \cap n_q$$

3) If $\alpha, \beta \in X_q$ with $\alpha < \beta$, then:

$$A^p_\alpha \setminus n_q \subseteq A^p_\beta \quad \text{and} \quad B^p_\alpha \setminus n_q \subseteq B^p_\beta$$

We will now define $\langle N_k \mid k \in \omega \rangle$ and $\langle p_F \mid F \in \mathcal{F} \rangle$ such that:

1) $\langle N_k \rangle_{k \in \omega}$ is an increasing sequence of natural numbers

2) If $F \in \mathcal{F}_k$, then:

a) $\mathrm{dom}(p_F) = F$

b) $\mathrm{ht}(p_F) = N_k$

2) If $F \in \mathcal{F}_\kappa$, then:

a) $\mathrm{dom}(p_F) \supseteq F$

b) $\mathrm{ht}(p_F) = N_\kappa$

For convenience, define:

$$A_\alpha^F = A_\alpha^{p_F} \quad \text{and} \quad B_\alpha^F = B_\alpha^{p_F}$$

For $\alpha \in F$

3) If $F, E \in \mathcal{F}$ and $E \subseteq F$,

then $p_F \leq p_E$

(If F is bigger than E, then p_F is a better aproximation than p_E)

4) Let $E, F \in \mathcal{F}_k$,

a) p_E and p_F are compatible

(there is $r \in \mathbb{P}$ with $r \leq p_E, p_F$)

4) Let $E, F \in \mathcal{F}_K$,

a) p_E and p_F are compatible

b) If $\alpha \in F$ and $\varphi = \varphi_{FE}$, then:

$$A_\alpha^F = A_{\varphi(\alpha)}^E$$

$$B_\alpha^F = B_{\varphi(\alpha)}^F$$

1) $\langle N_k \rangle_{k\in\omega}$ is an increasing sequence

2) If $F \in \mathcal{F}_{k}$, then: $\operatorname{dom}(p_F) \supseteq F$ and $ht(p_F) = N_k$

3) If $F, E \in \mathcal{F}$ and $E \subseteq F$, then $p_F \leq p_E$

4) Let $E, F \in \mathcal{F}_k$,

a) p_E and p_F are compatible

b) If $\alpha \in F$ and $\varphi = \varphi_{FE}$, then:

$A_\alpha^F = A_{\varphi(\alpha)}^E$ and $B_\alpha^F = B_{\varphi(\alpha)}^F$

We define this items recursively

base case: $k=0$

$\boxed{\text{base case: } k=0}$

Let $N_0 = 2$.

$\boxed{\text{base case: } k=0}$

Let $N_0 = 2$.

Choose $F \in \mathcal{F}_1$, say $F = \{\alpha\}$. Define:

$$p_F = (\{\alpha\}, 2, \{A_\alpha^F\}, \{B_\alpha^F\}) \quad \text{where:}$$

$\boxed{\text{base case: } k=0}$

Let $N_0 = 2$.

Choose $F \in \mathcal{F}_1$, say $F = \{\alpha\}$. Define:

$$p_F = (\{\alpha\}, 2, \{A_\alpha^F\}, \{B_\alpha^F\}) \quad \text{where:}$$

$$A_\alpha^F = \{0\} \quad \text{and} \quad B_\alpha^F = \{1\}$$

recursive step:

Assume we defined N_k and p_E for $E \in \mathcal{F}_k$.

We will now define N_{k+1} and p_F for $F \in \mathcal{F}_{k+1}$

Let $N_{k+1} = N_k + 5$.

Pick $F \in \mathcal{F}_{k+1}$. We know there are $F_0, F_1 \in \mathcal{F}_k$ and $R(F)$ such that:

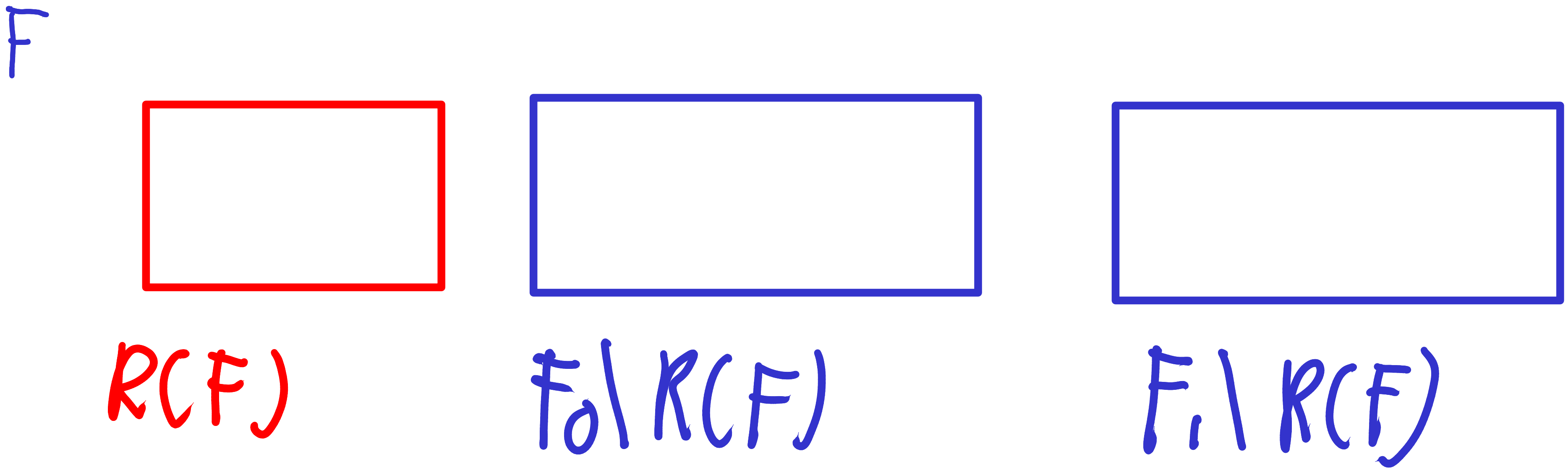
1) $F = F_0 \cup F_1$

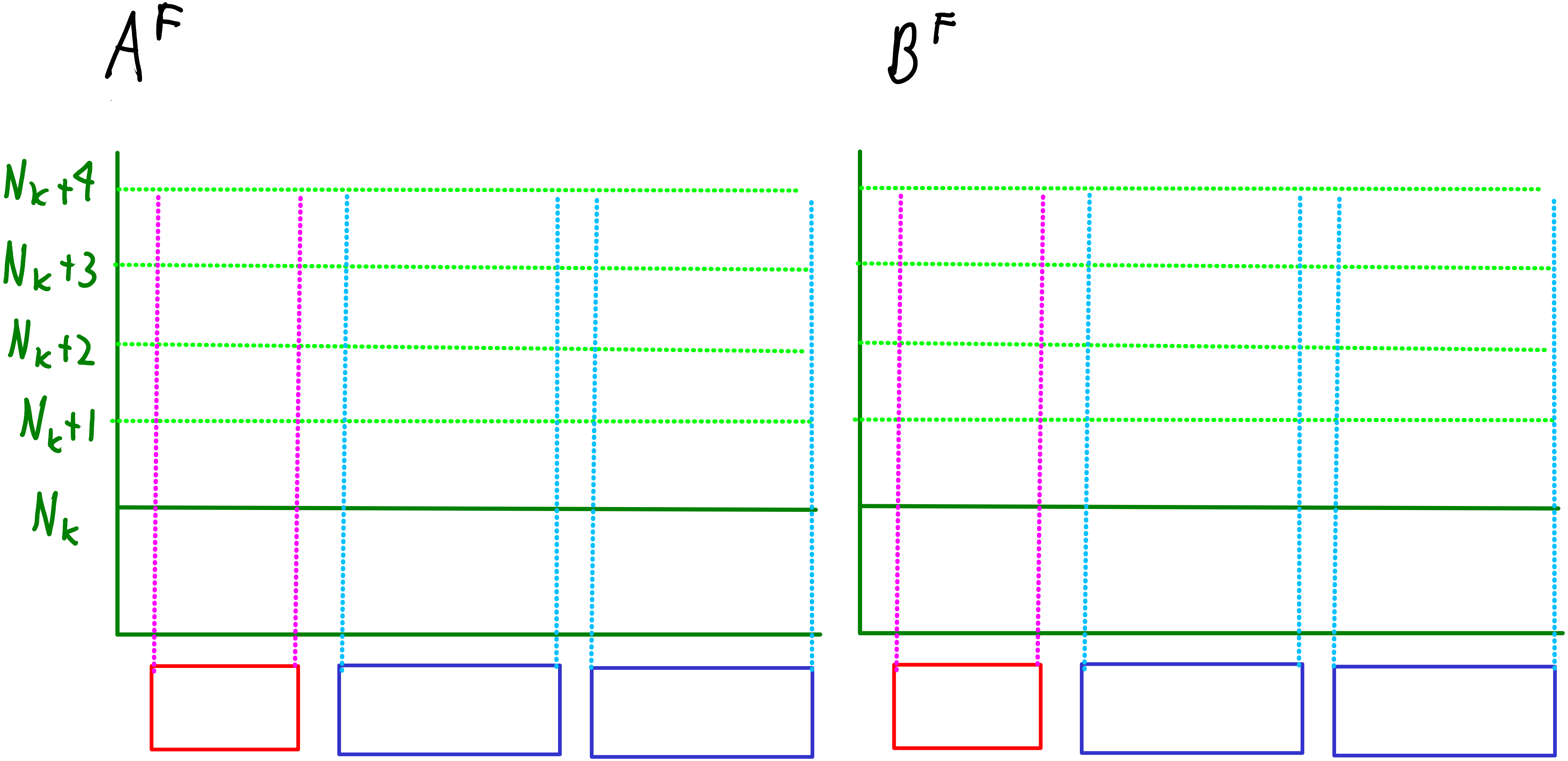
2) $F_0 \cap F_1 = R(F)$

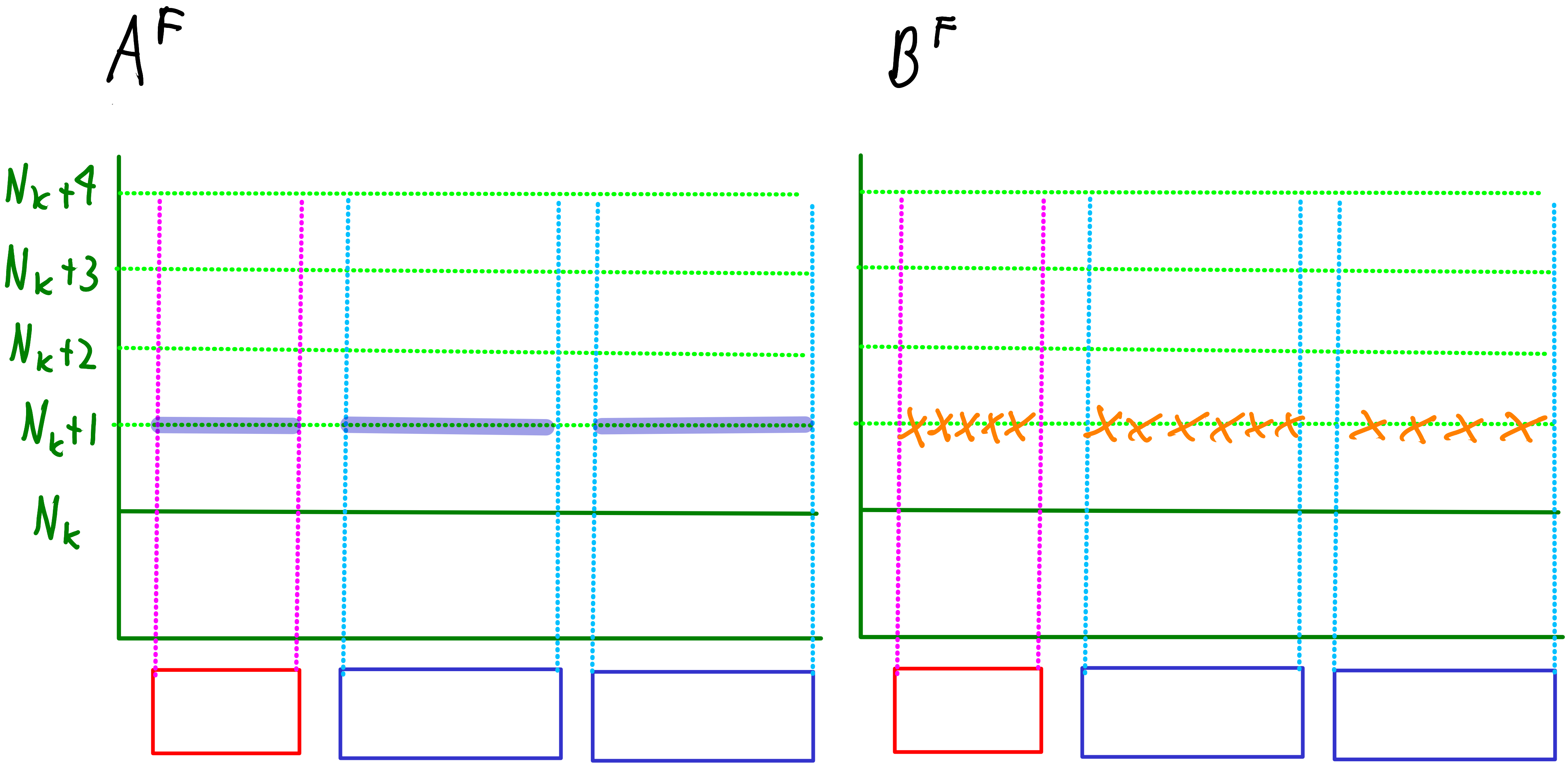
3) $|R(F)| = r_{k+1}$

4) $R(F) < F_0 \setminus R(F) < F_1 \setminus R(F)$

Pick $F \in \mathcal{F}_{k+1}$. We know there are $F_0, F_1 \in \mathcal{F}_k$ and $R(F)$ such that:




A^F
B^F
N_k+4
N_k+3
N_k+2
N_k+1
N_k



A^F
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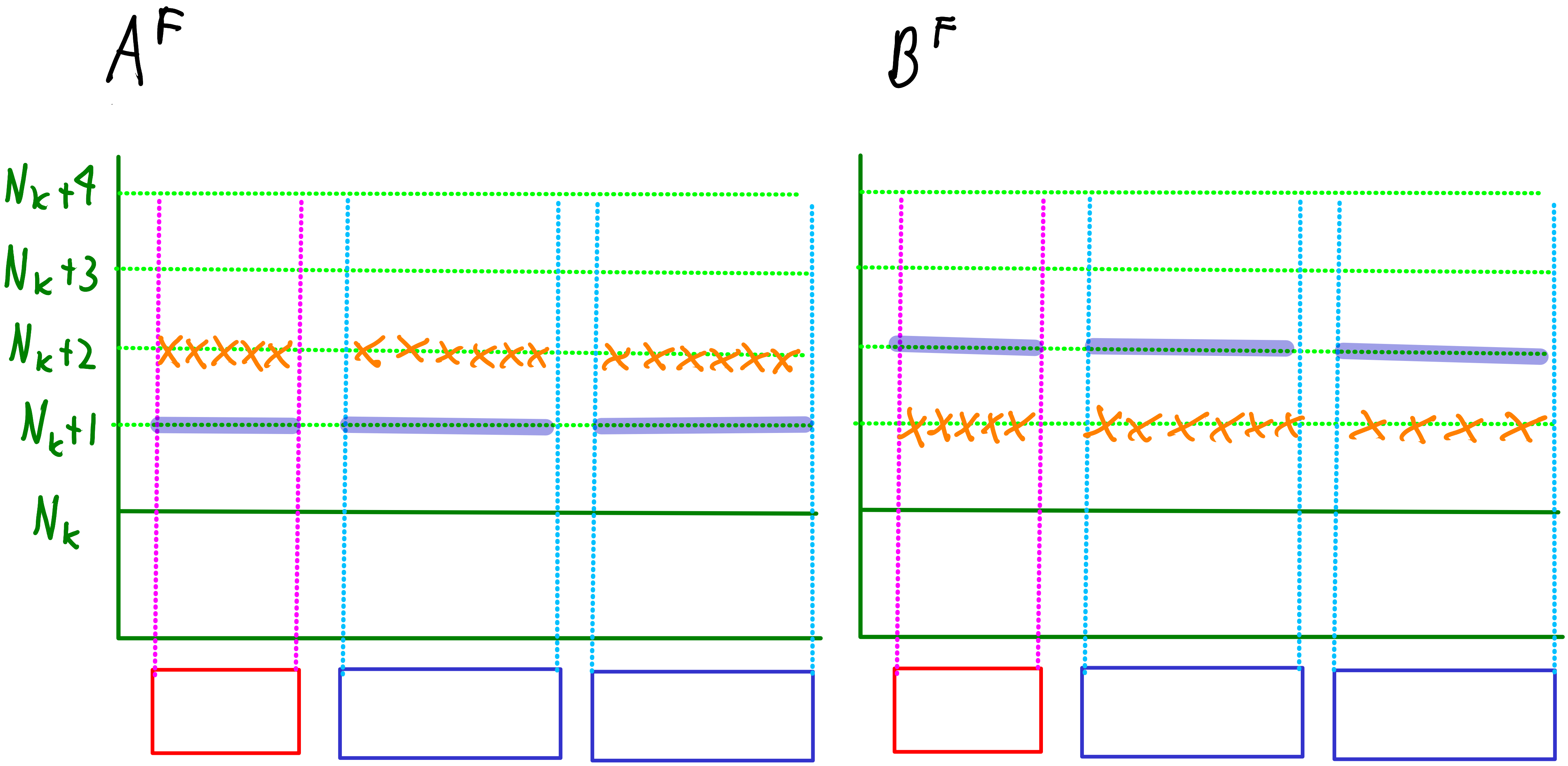

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N_k+1
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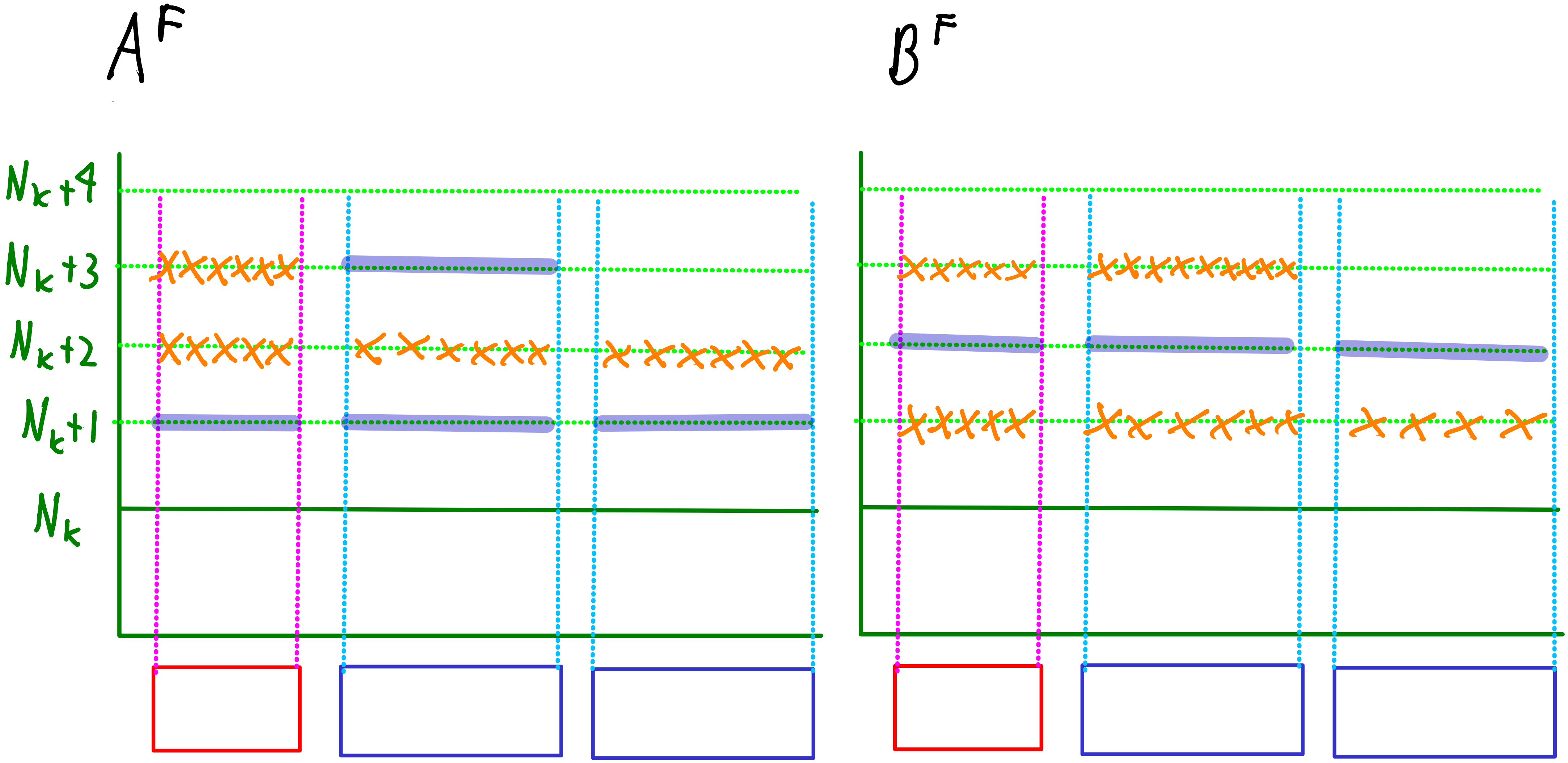

A^F
B^F
N_k+4
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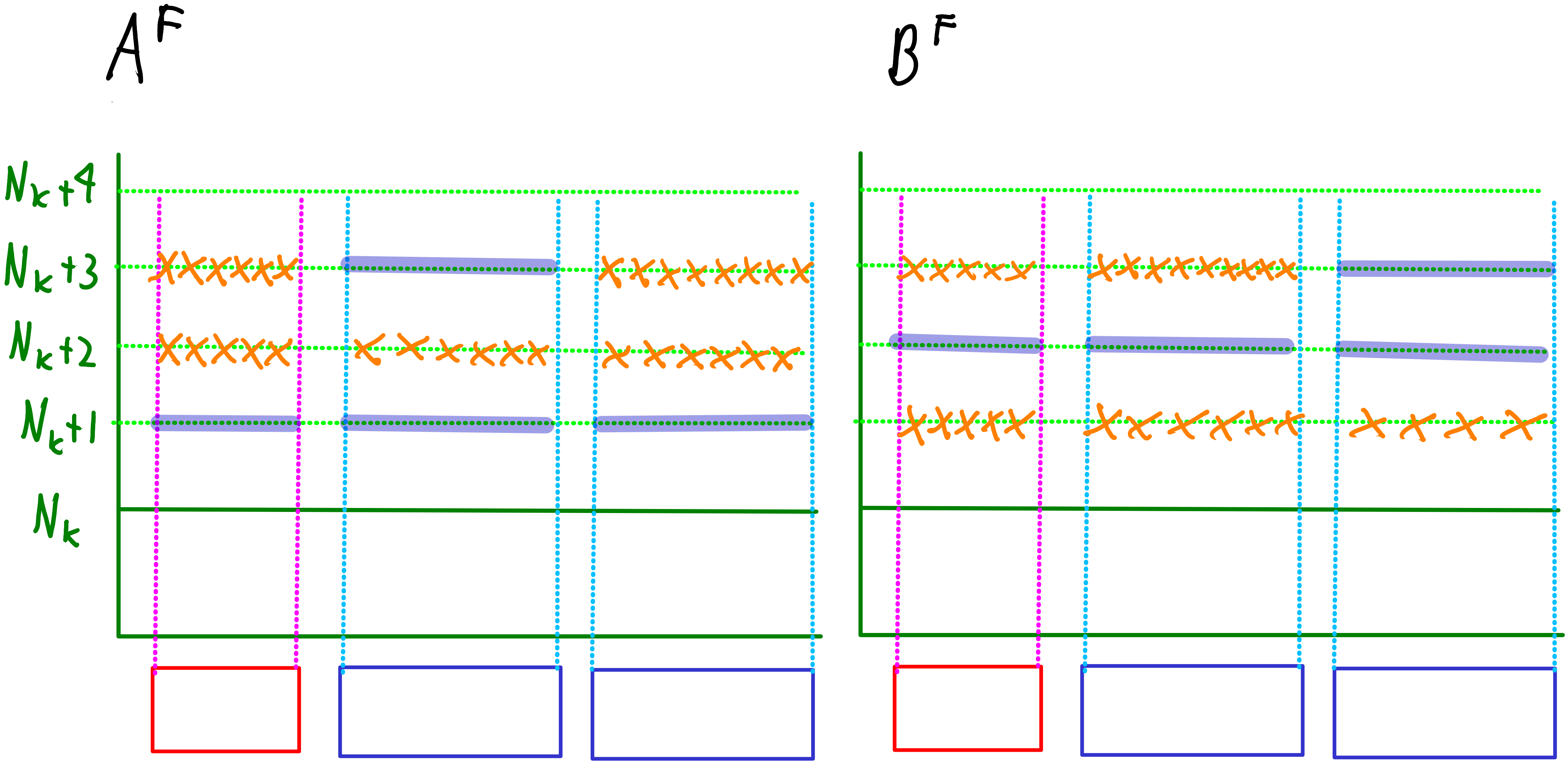
N_k



A^F
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N_k+4
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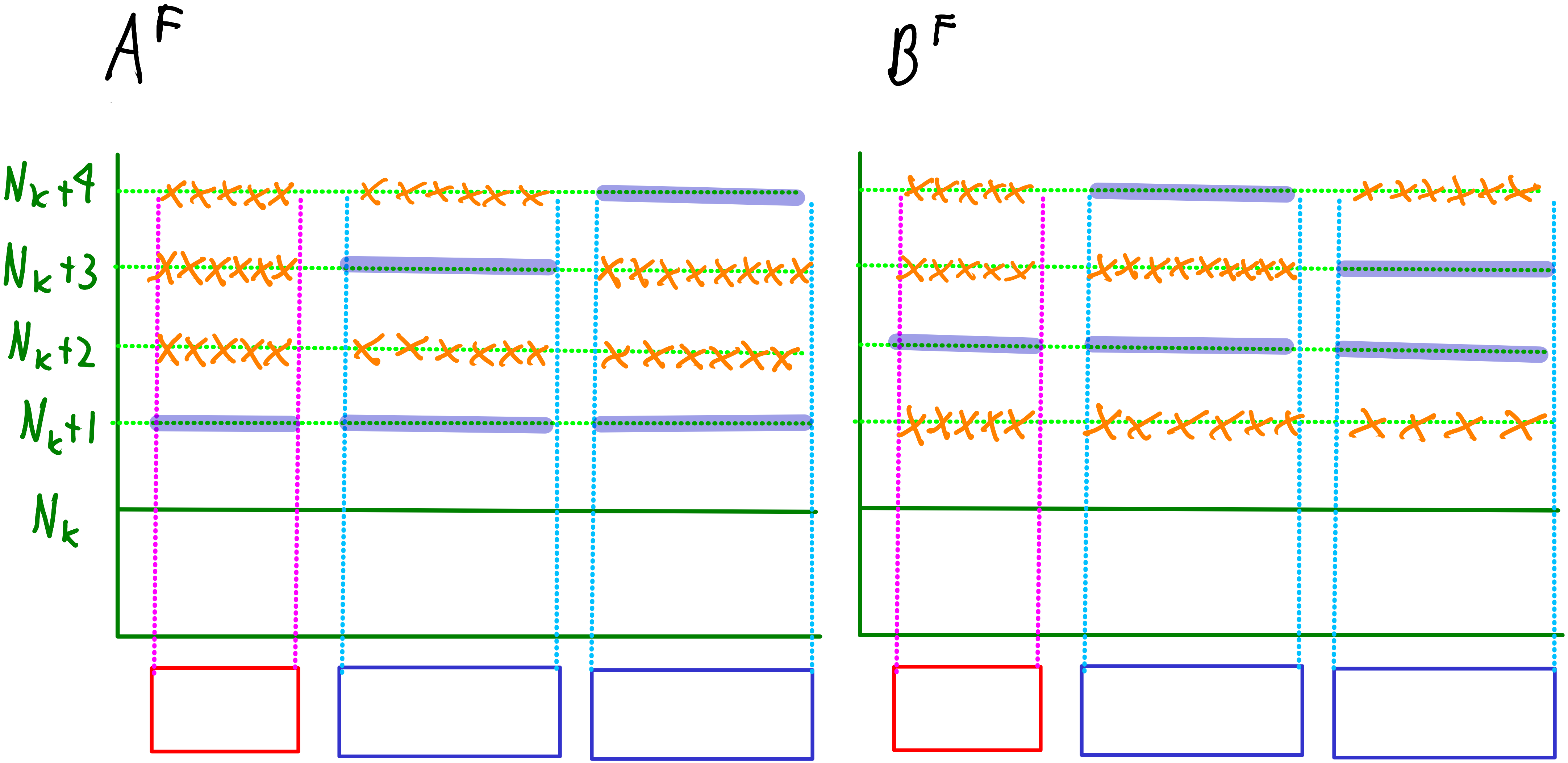

A^F
B^F
N_k+4
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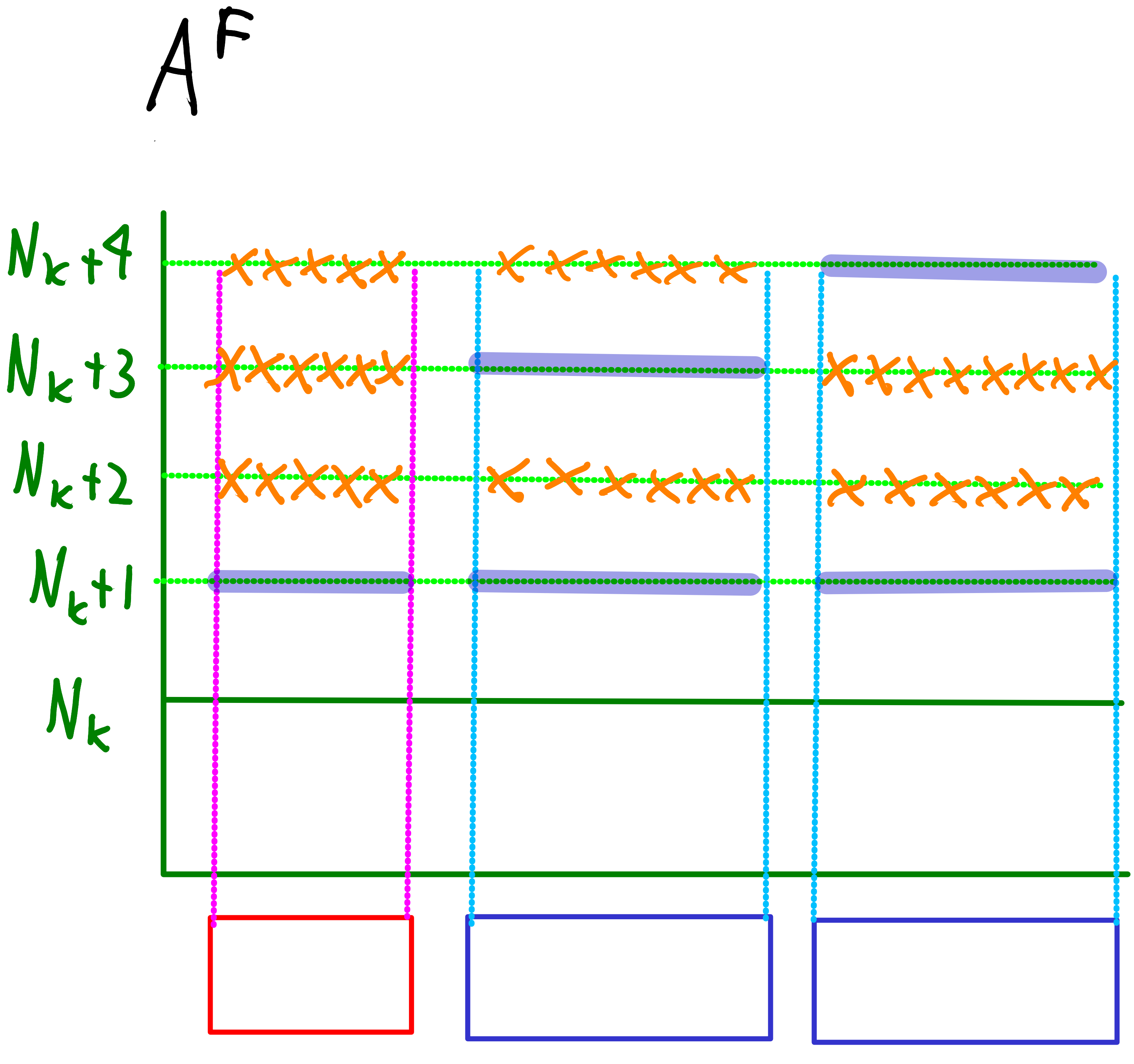
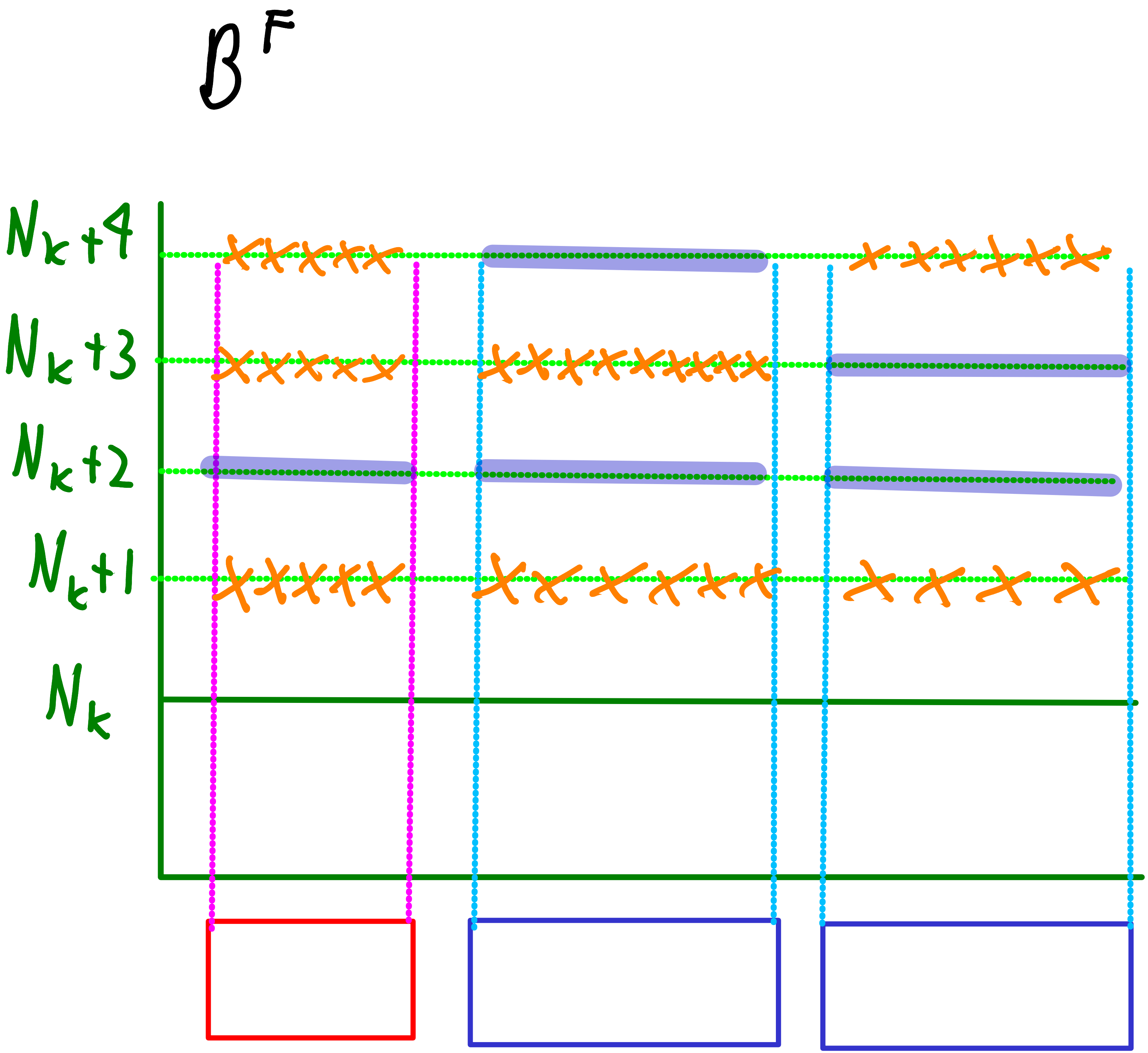
N_k



A^F
N_k+4
N_k+3
N_k+2
N_k+1
N_k



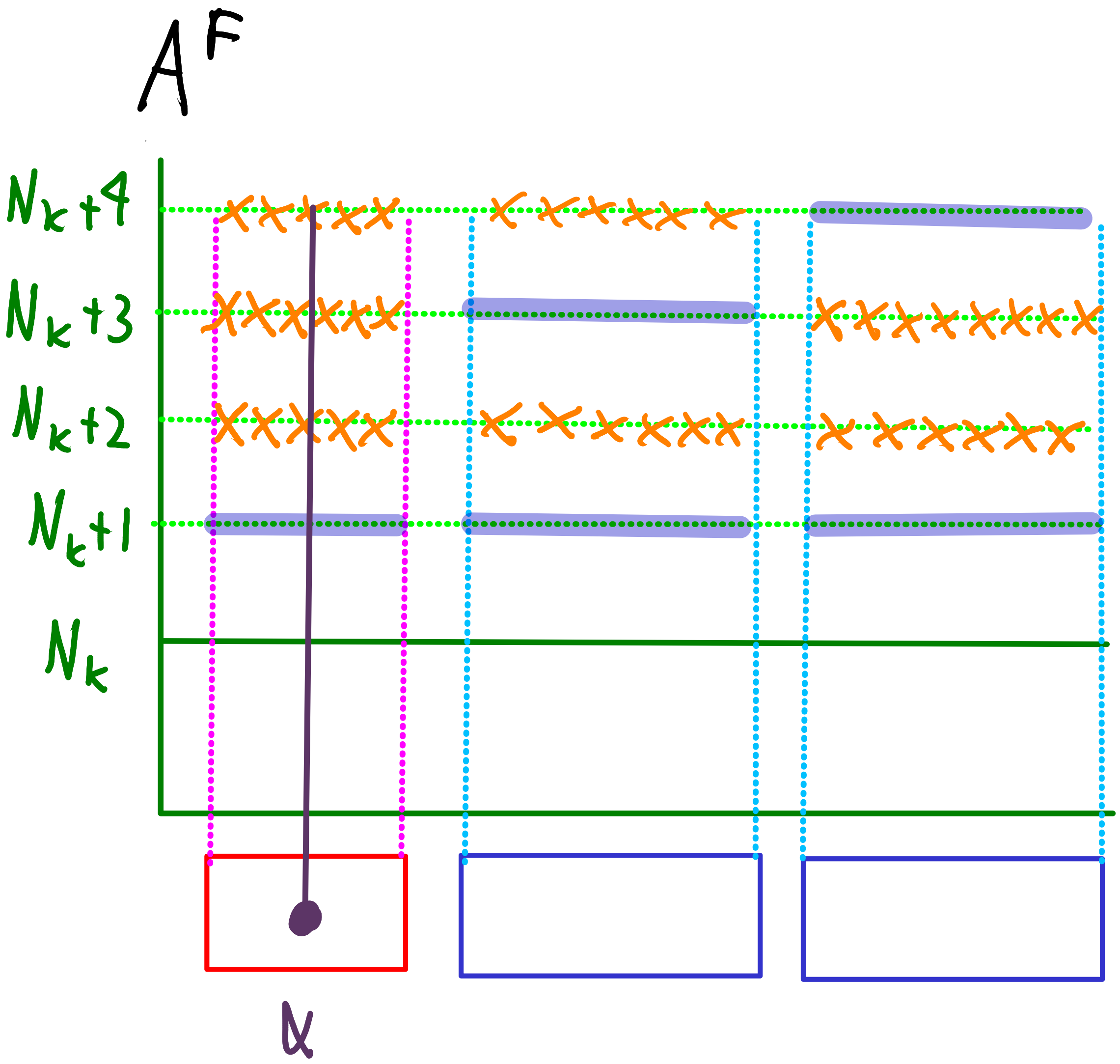
B^F
N_k+4
N_k+3
N_k+2
N_k+1
N_k

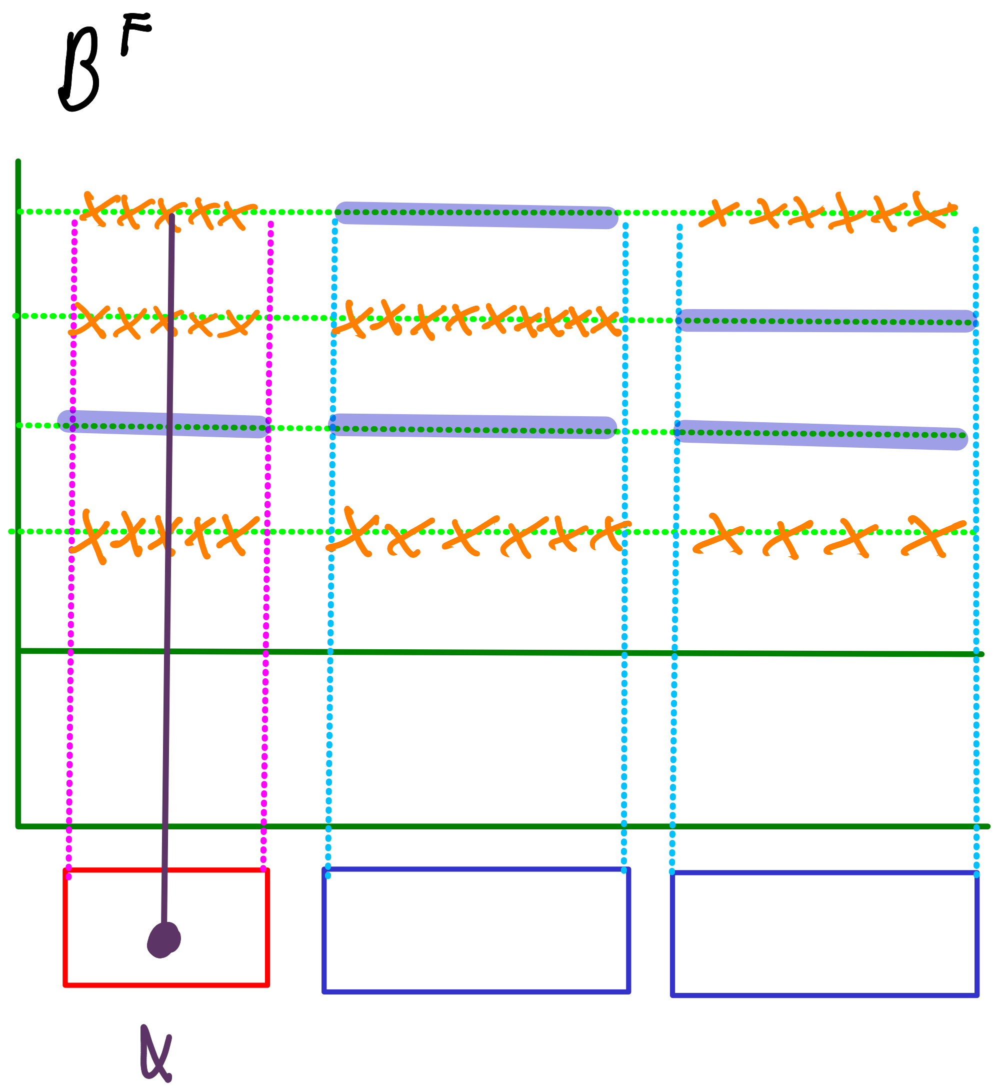
In words:

1) If $\alpha \in R(F)$

$$A_\alpha^F = A_\alpha^{F_0} \cup \{N_k + 1\}$$

$$B_\alpha^F = B_\alpha^{F_0} \cup \{N_{k+2}\}$$





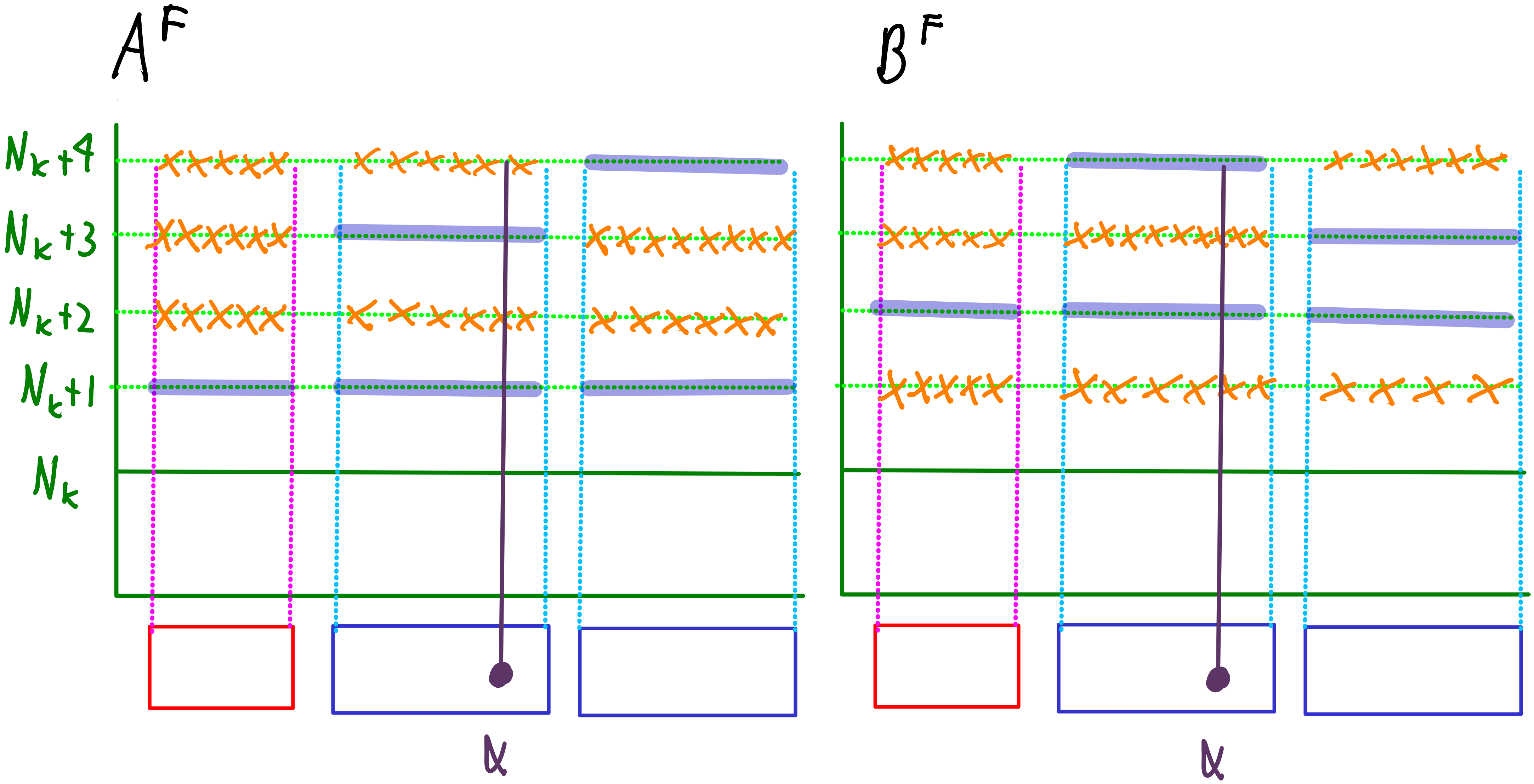
$$A^F_\alpha = A^{F_0}_\alpha \cup \{N_k + 1\}$$

$$B^F_\alpha = B^{F_0}_\alpha \cup \{N_k + 2\}$$

2) If $\alpha \in F_0 \setminus R(F)$

$$A_\alpha^F = A_\alpha^{F_0} \cup \{N_k+1, N_k+3\}$$

$$B_\alpha^F = B_\alpha^{F_0} \cup \{N_{k+2}, N_k+4\}$$



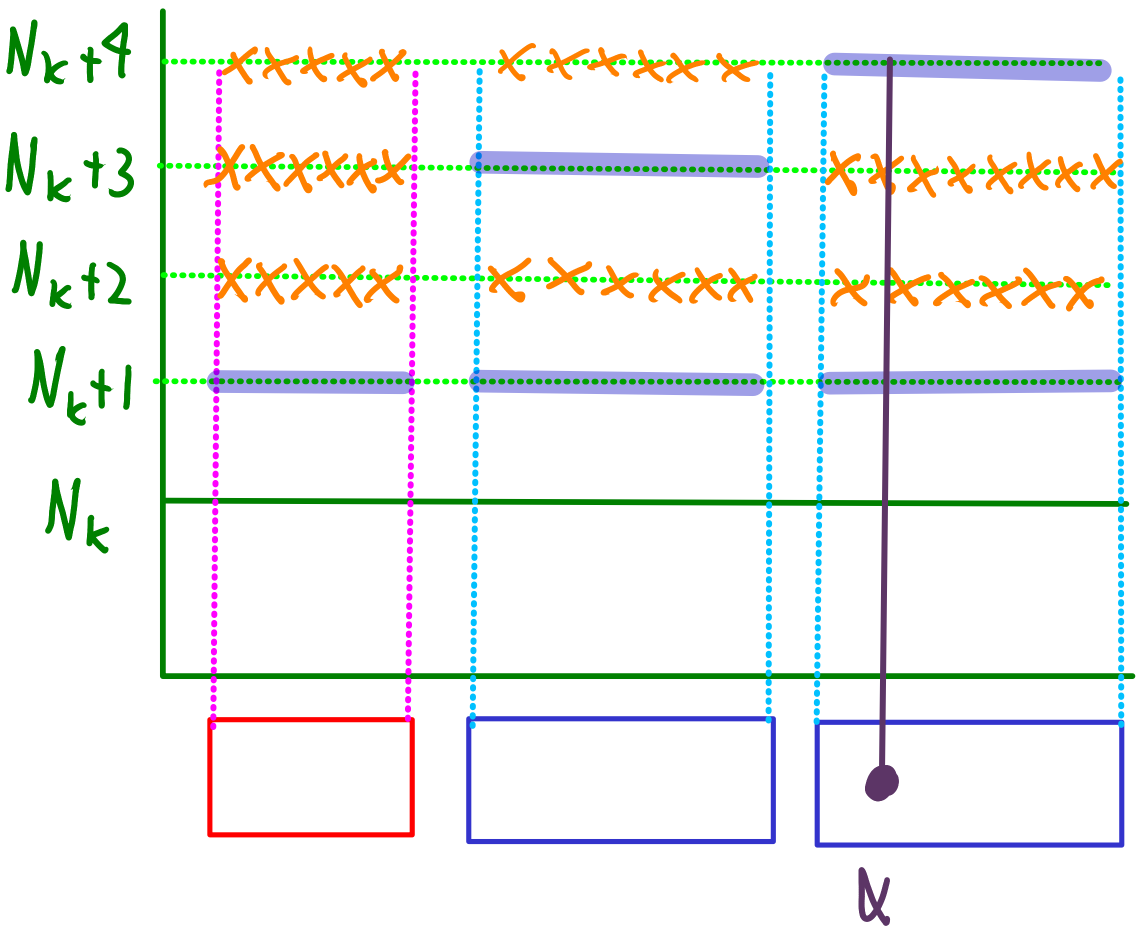
$$A_\alpha^F = A_\alpha^{F_0} \cup \{N_k+1, N_k+3\}$$
$$B_\alpha^F = B_\alpha^{F_0} \cup \{N_k+2, N_k+4\}$$

3) If $\alpha \in F_1 \setminus R(F)$

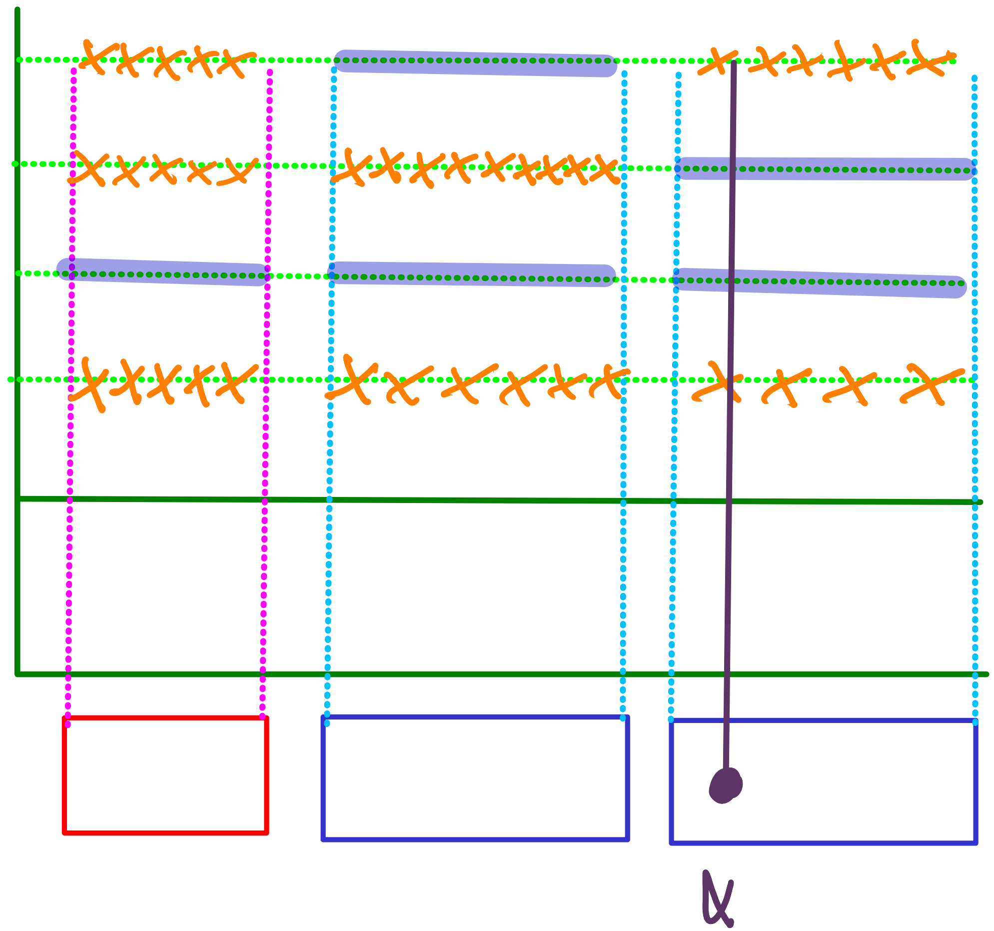
$$A_\alpha^F = A_\alpha^{F_0} \cup \{N_{k+1}, N_{k+4}\}$$

$$B_\alpha^F = B_\alpha^{F_0} \cup \{N_{k+2}, N_{k+3}\}$$

A^F



B^F

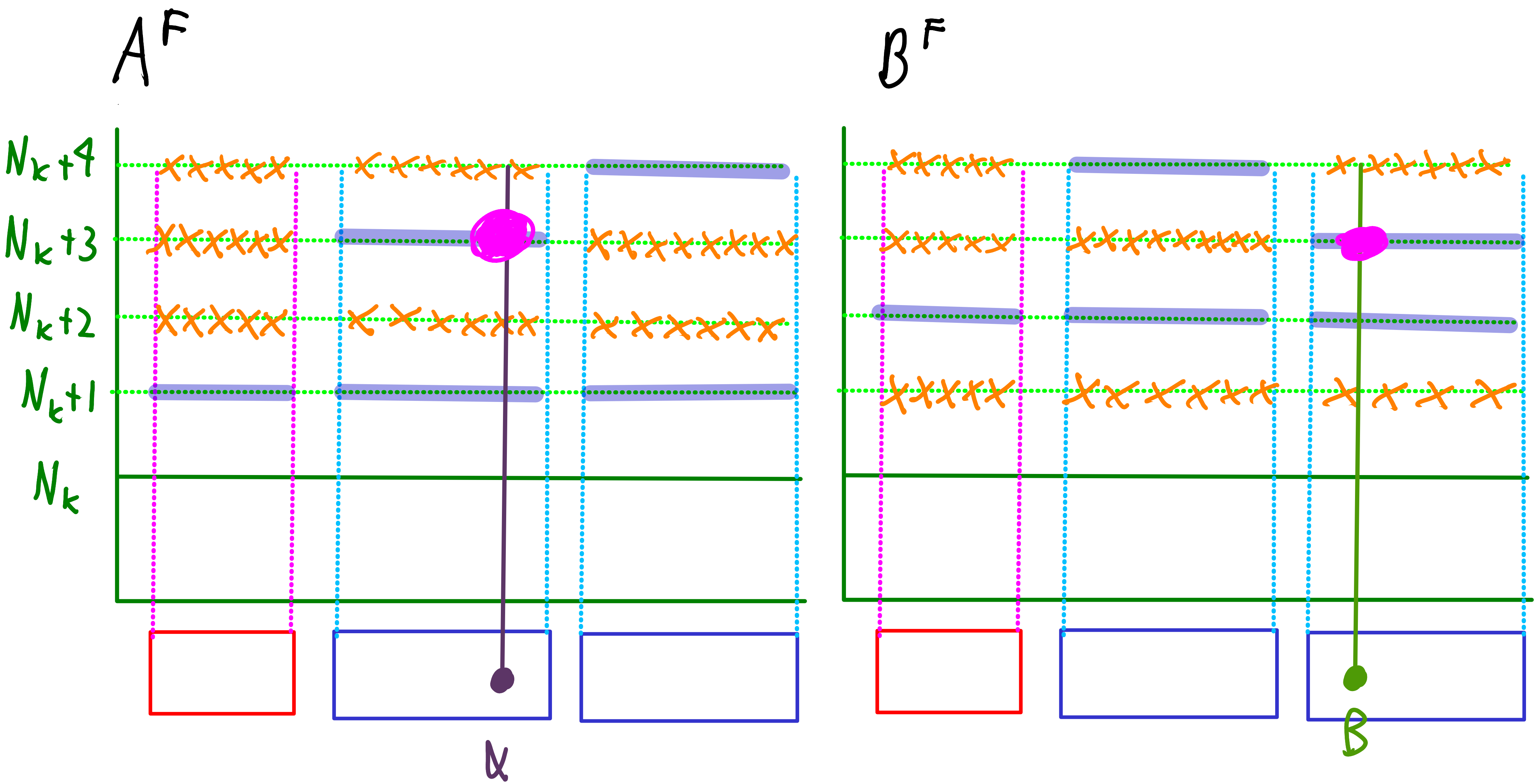


$$A^F_\alpha = A^{F_0}_\alpha \cup \{N_{k+1}, N_{k+4}\} \qquad B^F_\alpha = B^{F_0}_\alpha \cup \{N_{k+2}, N_{k+3}\}$$

The point is that if:

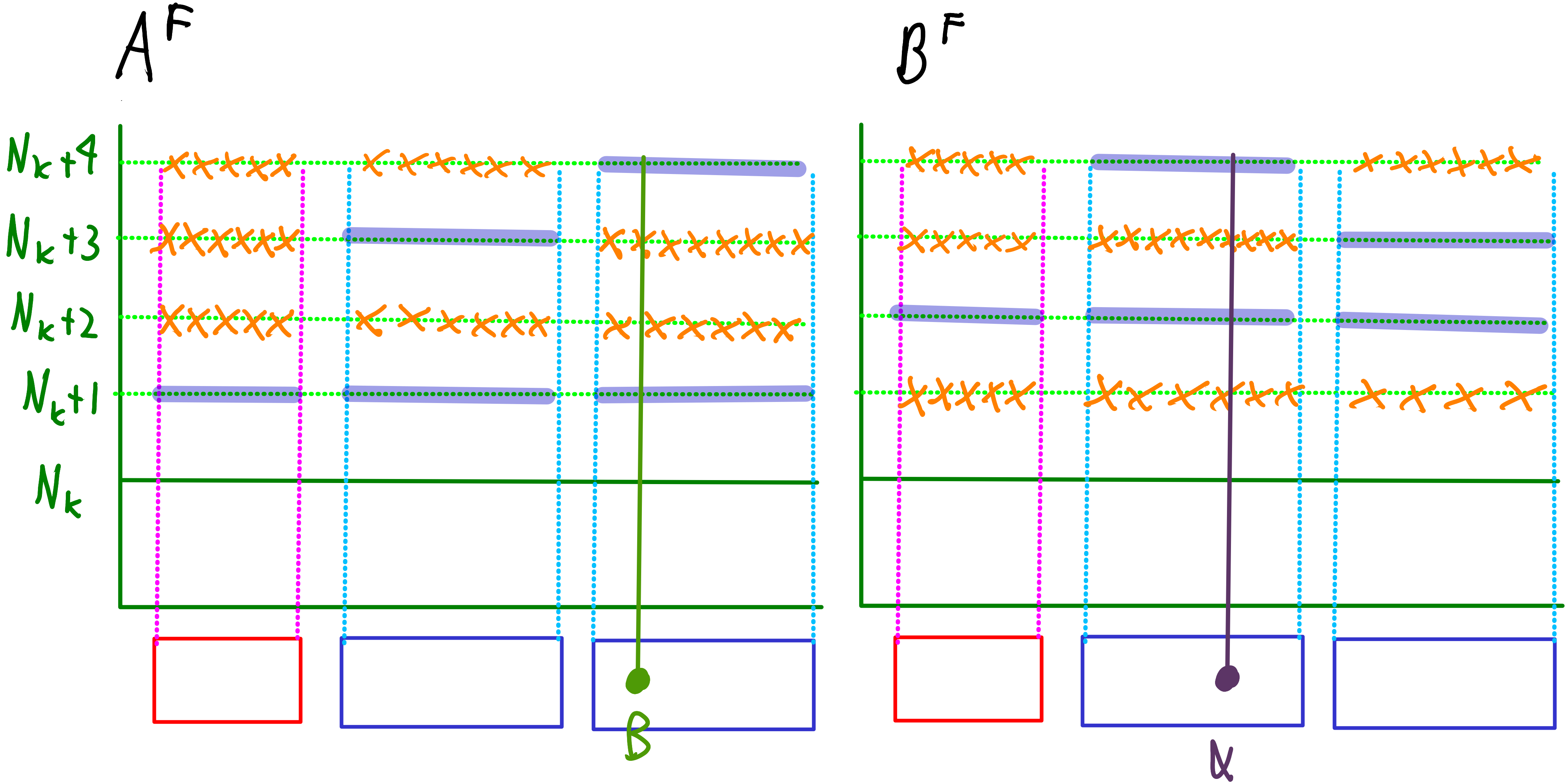
$$\alpha \in F_0 \setminus R(F) \quad \text{and} \quad \beta \in F_1 \setminus R(F)$$

Then:



$$A_\alpha^F = A_\alpha^{F_0} \cup \{N_k+1, N_k+3\} \qquad B_\beta^F = B_\beta^{F_0} \cup \{N_{k+2}, N_{k+3}\}$$

$$N_k+3 \in A_\alpha^F \cap B_\beta^F$$



$$A^F_\beta = A^{F_1}_\beta \cup \{N_{k+1}, N_{k+4}\} \qquad B^F_\alpha = B^{F_1}_\alpha \cup \{N_k+2, N_k+4\}$$

$$N_k+4 \in A^F_\beta \cap B^F_\alpha$$

This finishes the recursion construction

Now, for every $\alpha < \omega_1$ define:

$$A_\alpha = \bigcup \{ A_\alpha^F \mid \alpha \in F \in \mathcal{F} \}$$

$$B_\alpha = \bigcup \{ B_\alpha^F \mid \alpha \in F \in \mathcal{F} \}$$

Exercise

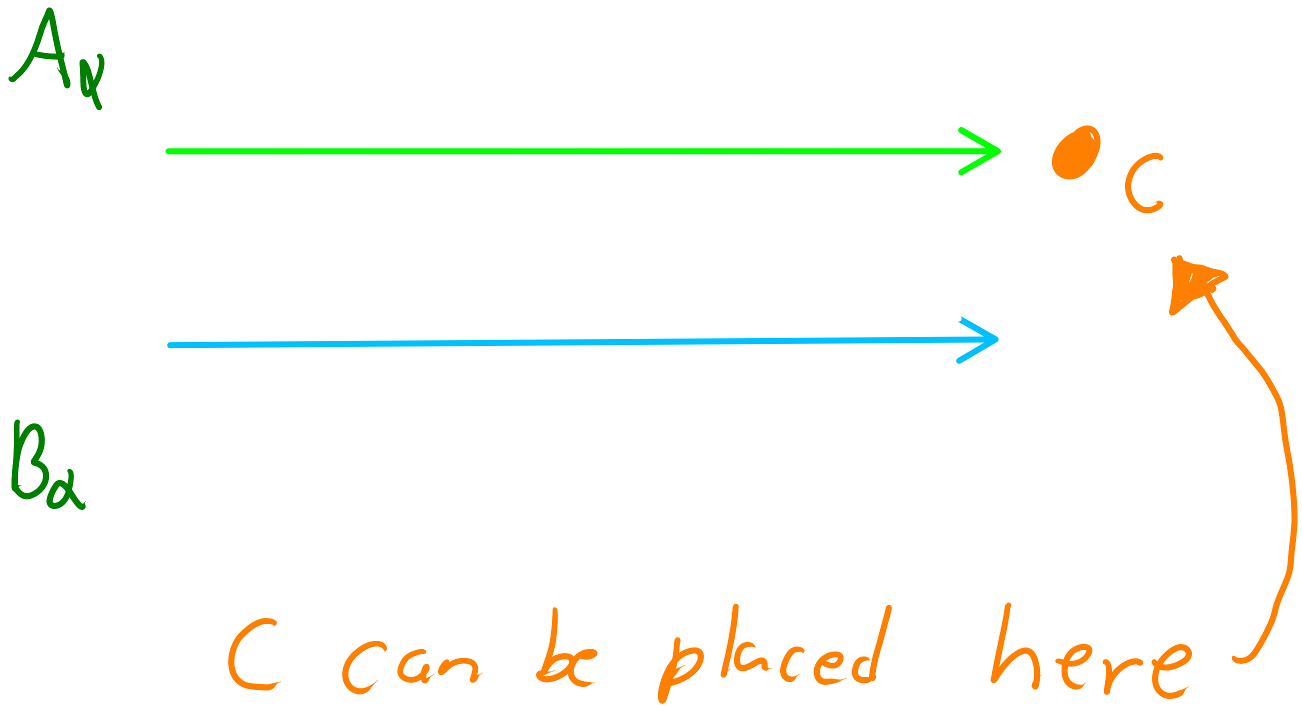
Prove that $(\langle A_\alpha \rangle, \langle B_\alpha \rangle)$ is a pregap

We now prove it is a gap. Assume there is $C \subseteq \omega$ such that for every $\alpha < \omega_1$:

1) $A_\alpha \subseteq^* C$

2) $B_\alpha \cap C$ is finite

We now prove it is a gap. Assume there is $C \subseteq \omega$ such that:



For every $\alpha < \omega_1$, we can find $l_\alpha \in \omega$ such that:

$$A_\alpha \setminus l_\alpha \subseteq C$$

$$B_\alpha \setminus l_\alpha \subseteq \omega \setminus C$$

By a counting argument, we can find $\alpha < \beta$ and $l < \omega$ such that:

1) $l = l_\alpha = l_\beta$

2) $A_\alpha \cap l = A_\beta \cap l$

$B_\alpha \cap l = B_\beta \cap l$

Now, let $k \in \omega$ be the first such that there is $F \in \mathcal{F}$ such that

$$\alpha, \beta \in F$$

Now, let $k \in \omega$ be the first such that there is $F \in \mathcal{F}$ such that

$$\alpha, \beta \in F$$

(In other words, k is the "distance" between α and β, as defined in the previous lecture)

Claim

$\alpha \in F_0 \setminus R(F)$ and $\beta \in F_1 \setminus R(F)$

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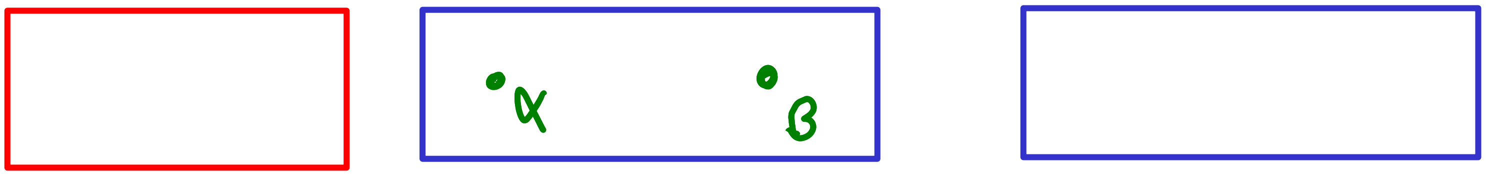
If $\beta \in F_0$ then:



α will also be in F_0 ($\alpha < \beta$), but this contradicts the minimality of k

If $\alpha \in F_1$ then:

β will also be in F_1 ($\alpha < \beta$), but this contradicts the minimality of k

This finishes the proof of the claim

By the construction, we get that there is $i \in A_\alpha \cap B_\beta$.

claim

$\ell \leq i$

Claim

$\ell \leq i$

Recall:

$i \in A_\alpha \cap B_\beta$

$A_\alpha \cap \ell = A_\alpha \cap \ell$

Assume $i < \ell$. Now:

$$i \in A_\alpha \cap \ell = A_\beta \cap \ell$$

$$\Rightarrow i \in A_\beta \cap B_\beta$$

which is a contradiction

Finally, recall that:

$$A_\alpha \setminus \ell \subseteq C$$

$$B_\beta \setminus \ell \subseteq \omega \setminus C$$

so

$$i \in A_\alpha \setminus \ell \Rightarrow i \in C$$

$$i \in B_\beta \setminus \ell \Rightarrow i \in \omega \setminus C$$

which is obviously a contradiction

$\square$

Constructing an Aronszajn tree

Def

An **Aronszajn tree** is a tree such that:

1) It has height ω_1

2) Its levels are countable

3) It has no cofinal branches